

ON THE QUATERNIONIC MANIFOLDS WHOSE TWISTOR SPACES ARE FANO MANIFOLDS

RADU PANTILIE

ABSTRACT. Let M be a quaternionic manifold, $\dim M = 4k$, whose twistor space is a Fano manifold. We prove the following:

- (a) M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ if and only if $M = \mathbb{H}P^k$,
- (b) either $b_2(M) = 0$ or $M = \mathrm{Gr}_2(k+2, \mathbb{C})$.

This generalizes results of S. Salamon and C. R. LeBrun, respectively, who obtained the same conclusions under the assumption that M is a complete quaternionic-Kähler manifold with positive scalar curvature.

1. INTRODUCTION

An *almost quaternionic structure* on a manifold M is a reduction of its frame bundle to $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H})$. Then the obstruction for M to admit a ‘reduction’ to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ is an element of $H^2(M, \mathbb{Z}_2)$ [8]. Equivalently, this is the second Stiefel-Whitney class of the oriented Riemannian vector bundle Q induced by the Lie groups morphism $\mathrm{Sp}(1) \cdot \mathrm{GL}(k, \mathbb{H}) \rightarrow \mathrm{SO}(3)$, $\pm(a, A) \mapsto \pm a$.

If $\dim M \geq 8$ then the almost quaternionic structure is *integrable* if there exists a torsion free connection on M which is compatible (with the structural group) [12]. Equivalently (see [3]), there exists a compatible connection ∇ on M such that the almost complex structure induced by ∇ on the sphere bundle Z of Q is integrable. Then the complex manifold Z is the *twistor space* of M and the fibres of $\pi : Z \rightarrow M$ are the ‘real’ *twistor lines*; furthermore, Z is endowed with a conjugation (given by the antipodal map on the fibres of π). Conversely, Z together with its conjugation and a real twistor line determines M (see [9]). Furthermore, by [12] and [10], there exists a holomorphic line bundle \mathcal{L} over Z whose restriction to any twistor line has Chern number 2. It follows quickly that M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$ if and only if \mathcal{L} admits a square root.

Further natural restrictions can be obtained by assuming that there exists a Riemannian metric on M for which the holonomy group of its Levi-Civita connection is contained by $\mathrm{Sp}(1) \cdot \mathrm{Sp}(k)$; then M is called *quaternionic-Kähler*. It follows [11] that any quaternionic-Kähler manifold is an Einstein manifold, and, assuming, further, completeness and the scalar curvature positive, the corresponding twistor space is a Fano

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manifold. Also, by [11, Theorem 6.3], $\mathbb{H}P^k$ is the only such quaternionic-Kähler manifold which admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$.

Another result, in the same vein, is [7] that for any complete quaternionic-Kähler manifold M with positive scalar curvature we have that either its second Betti number $b_2(M)$ is zero, or M is the Grassmannian $\mathrm{Gr}_2(k+2, \mathbb{C})$, where $\dim M = 4k$.

In this paper, we generalize these two results of [11] and [7], respectively, to the class of quaternionic manifolds whose twistor spaces are Fano manifolds.

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2. THE RESULTS

As the four-dimensional case was elucidated in [2], we consider only quaternionic manifolds of dimension at least 8.

The following result generalizes [11, Theorem 6.3].

Theorem 2.1. *Let M be a quaternionic manifold, $\dim M = 4k \geq 8$, which admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$; denote by Z the twistor space of M .*

Then the following assertions are equivalent:

- (i) $M = \mathbb{H}P^k$;
- (ii) M is simply-connected, $b_2(M) = 0$ and Z is projective (that is, Z can be embedded as a compact complex submanifold of a complex projective space);
- (iii) Z is a Fano manifold (that is, Z is compact and its anticanonical line bundle is ample).

Proof. It is obvious that if (i) holds then both (ii) and (iii) are satisfied, as $Z = \mathbb{C}P^{2k+1}$ and $M = \mathbb{H}P^k$.

Further, as the restriction of the holomorphic cotangent bundle to each twistor line is $\mathcal{O}(-2) \oplus 2k\mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the tautological line bundle, essentially the same proof as for [2, Proposition 2.2(ii)] implies that any holomorphic form of positive degree on Z is zero. Consequently, if Z is projective, from the exact sequence of cohomology groups associated to the exact sequence of complex Lie groups $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\} \rightarrow 0$ (determined by the exponential) we deduce that the Picard group $\mathrm{Pic}(Z)$ is isomorphic to $H^2(Z, \mathbb{Z})$. Furthermore, if (ii) holds then, also, Z is simply-connected (by the homotopy exact sequence determined by the smooth bundle $Z \rightarrow M$), and, hence, $\mathrm{Pic}(Z)$ has no torsion. Also, as $b_2(Z) = b_2(M) + 1$ (see [7]), $\mathrm{Pic}(Z)$ has rank 1. We have, thus, proved that $\mathrm{Pic}(Z)$ is isomorphic to \mathbb{Z} .

Let \mathcal{L} be the restriction to Z of the dual of the tautological line bundle over the complex projective space in which Z is embedded. As both the restriction of \mathcal{L} and of the anticanonical line bundle K_Z^* of Z , to a twistor line, are positive we deduce that $(K_Z^*)^p = \mathcal{L}^q$, for some positive integers p and q . Thus, also $(K_Z^*)^p$ is very ample, and

(ii) \implies (iii) is proved.

To complete the proof it is sufficient to show that (iii) \implies (i). We claim that, if (iii) holds, there exists a holomorphic line bundle \mathcal{L} over Z such that:

- (a) \mathcal{L} is ample;
- (b) \mathcal{L} restricted to each twistor line is (isomorphic to) $\mathcal{O}(1)$.

Indeed, from the assumption that M admits a reduction to $\mathrm{Sp}(1) \times \mathrm{GL}(k, \mathbb{H})$, by [12] and [10] there exists a holomorphic line bundle \mathcal{L}_1 over Z which satisfies condition (b), above; moreover, \mathcal{L}_1 is endowed with a morphism of (real) vector bundles whose square is -1 and which is an anti-holomorphic diffeomorphism covering the conjugation of Z (given, on each fibre of $Z \rightarrow M$, by the antipodal map). We shall show that after tensorising, if necessary, \mathcal{L}_1 with a holomorphic line bundle, whose restriction to each twistor line is trivial, we obtain a line bundle satisfying (a).

For this, firstly, note that $K_Z^* (= \Lambda_{\mathbb{C}}^{2k+1} T^*Z)$ restricted to each twistor line is $\mathcal{O}(2k+2)$. Hence, $K_Z \otimes \mathcal{L}_1^{2k+2}$ restricted to each twistor line is trivial; moreover, this holomorphic line bundle is endowed with a conjugation (that is, an involutive morphism of vector bundles which is an anti-holomorphic diffeomorphism) covering the conjugation of Z . Therefore $K_Z \otimes \mathcal{L}_1^{2k+2}$ corresponds, through the Ward transform, to a (real) line bundle L over M endowed with an anti-self-dual connection (that is, a connection whose curvature form is such that its $(0,2)$ -part, with respect to any admissible linear complex structure on M , is zero).

As M is simply-connected (because Z is Fano and therefore simply-connected, and the fibres of the projection $Z \rightarrow M$ are connected), L is orientable and, hence, there exists a line bundle L_1 such that $L = L_1^{2k+2}$; furthermore, this isomorphism is connection preserving with respect to a unique anti-self-dual connection on L_1 . Hence, L_1 corresponds to a holomorphic line bundle \mathcal{L}_2 over Z whose restriction to each twistor line is trivial, and such that $K_Z \otimes \mathcal{L}_1^{2k+2} = \mathcal{L}_2^{2k+2}$.

Thus, since K_Z^* is ample, $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^*$ satisfies (a) and (b), above. Moreover, \mathcal{L} is endowed with a morphism of vector bundles τ whose square is -1 and which is an anti-holomorphic diffeomorphism covering the conjugation of Z . Hence, τ induces a linear complex structure J on $H^0(Z, \mathcal{L})$ which anti-commutes with its canonical complex structure.

By [4, Corollary 2.4], Z is a complex projective space and the twistor lines are just the complex projective lines; moreover, Z is the projectivisation of the dual of $H^0(Z, \mathcal{L})$. Furthermore, J induces on the dual E of $H^0(Z, \mathcal{L})$ a linear quaternionic structure with respect to which the fibres of $Z \rightarrow M$ are those complex projective lines obtained through the complex projectivisation of the quaternionic vector subspaces of E of real dimension 4. Thus, $Z = PE$, M is the quaternionic projective space $P_{\mathbb{H}}E$, and $Z \rightarrow M$ is the canonical projection $PE \rightarrow P_{\mathbb{H}}E$. The proof is complete. \square

The following result generalizes [7, Theorem 1].

Theorem 2.2. *Let M be a quaternionic manifold, $\dim M = 4k \geq 8$, whose twistor space is a Fano manifold.*

Then either $b_2(M) = 0$ or $M = \text{Gr}_2(k+2, \mathbb{C})$.

Proof. Let Z be the twistor space of M . Similarly to the proof of Theorem 2.1, we obtain a holomorphic line bundle \mathcal{L} over Z such that $\mathcal{L}^{k+1} = K_Z^*$. Furthermore, \mathcal{L} admits a square root if and only if M admits a reduction to $\text{Sp}(1) \times \text{GL}(k, \mathbb{H})$. Therefore, by Theorem 2.1, either $M = \mathbb{H}P^k$ or $k+1$ is the greatest natural number n for which K_Z^* admits a n -th root. From now on, in this proof, we shall assume that the latter holds.

Now, just like in the proof of [7, Theorem 1], by using [13], we obtain that if $b_2(M) \neq 0$ then (at least) one of the following three statements holds:

- (i) $Z = \mathbb{C}P^k \times Q_{k+1}$, where Q_{k+1} is the nondegenerate hyperquadric in $\mathbb{C}P^{k+2}$,
- (ii) Z is the projectivisation of the holomorphic cotangent bundle of $\mathbb{C}P^{k+1}$,
- (iii) Z is $\mathbb{C}P^{2k+1}$ blown up along $\mathbb{C}P^{k-1}$.

The fact that (i) cannot occur is a consequence of Proposition 2.3, below.

In the remaining two cases, it follows that M can be locally identified (through quaternionic diffeomorphisms) with $\text{Gr}_2(k+2, \mathbb{C})$ or with $\mathbb{H}P^k$, respectively. By using that M is compact and simply-connected, a standard argument shows that either $M = \text{Gr}_2(k+2, \mathbb{C})$ or $M = \mathbb{H}P^k$. As the latter leads to a contradiction, the proof is complete. \square

The following result, also interesting in itself, was used in the proof of Theorem 2.2.

Proposition 2.3 ([6]). *Let Q_{k+1} be the nondegenerate hyperquadric in $\mathbb{C}P^{k+2}$. Then no open subset of $\mathbb{C}P^k \times Q_{k+1}$ can be the twistor space of a quaternionic manifold.*

Proof. We shall prove that $Y = \mathbb{C}P^k \times Q_{k+1}$ does not admit an embedded Riemann sphere whose normal bundle is $2k\mathcal{O}(1)$. Indeed, let L_1 and L_2 be the restrictions to $\mathbb{C}P^k$ and Q_{k+1} of the duals of the tautological line bundles on $\mathbb{C}P^k$ and $\mathbb{C}P^{k+2}$, respectively. We have that both L_1 and L_2 are very ample and, also, $K_{\mathbb{C}P^k}^* = (L_1)^{k+1}$, $K_{Q_{k+1}}^* = (L_2)^{k+1}$ (for the latter, use the adjunction formula mentioned in [1, p. 147]). Thus, on denoting by π_1 and π_2 the projections from Y onto its factors, respectively, we obtain that, also, $L = \pi_1^* L_1 \otimes \pi_2^* L_2$ is very ample, and $K_Y^* = L^{k+1}$. Therefore if Y would admit an embedded Riemann sphere t whose normal bundle is $2k\mathcal{O}(1)$ then $L|_t = \mathcal{O}(2)$. On embedding Y into the projectivisation of the dual of $H^0(Y, L)$, we obtain that t has degree two and therefore it is a conic. It follows that any two points of Y are joined by a conic. But, according to [5], Y cannot have this property, thus completing the proof. \square

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E-mail address: `radu.pantilie@imar.ro`

R. PANTILIE, INSTITUTUL DE MATEMATICĂ “SIMION STOILOW” AL ACADEMIEI ROMÂNE, C.P. 1-764, 014700, BUCUREȘTI, ROMÂNIA